# DOCUMENT RESUME

ED 121 622 SE 020 770

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|---|---|
| AUTHOR | Schaaf, William L. |
| TITLE | A Bibliography of Recreational Mathematics, Volume 1. Fourth Edition. |
| INSTITUTION | National Council of Teachers of Mathematics, Inc., Reston, Va. |
| PUB DATE | 70 |
| NOTE | 160p.; For related documents see Ed 040 874 and ED 087 631 |
| AVAILABLE FROM | National Council of Teachers of Mathematics, Inc., 1906 Association Drive, Reston, Virginia 22091 |
| EDRS PRICE | MF-$0.83 Plus Postage. HC Not Available from EDRS. |
| DESCRIPTORS | *Annotated Bibliographies; *Games; Geometric Concepts; History; *Literature Guides; *Mathematical Enrichment; Mathematics; *Mathematics Education; Number Concepts; Reference Books |

ABSTRACT

This book is a partially annotated bibliography of books, articles, and periodicals concerned with mathematical games, puzzles, and amusements. It is a reprinting of Volume 1 of a three-volume series. This volume, originally published in 1955, treats problems and recreations which have been important in the history of mathematics as well as some of more modern invention. The book is intended for use by both professional and amateur mathematicians. Works on recreational mathematics are listed in eight broad categories: general works, arithmetic and algebraic recreations, geometric recreations, assorted recreations, magic squares, the Pythagorean relationship, famous problems of antiquity, and mathematical miscellanies. (SD)

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A BIBLIOGRAPHY OF

# recreational mathematics

## VOLUME I

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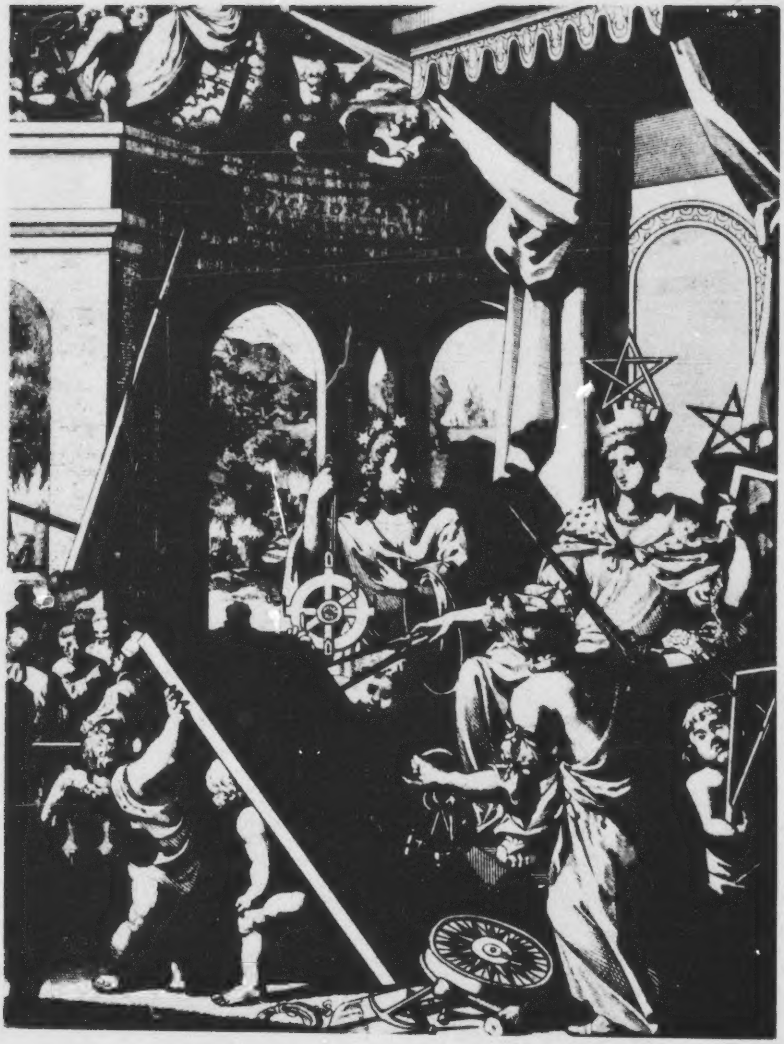

From J. Ozanam: *Dictionaire Mathématique.*
Amsterdam. 1691

### A BIBLIOGRAPHY OF
# recreational mathematics

### VOLUME
### I

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FOURTH EDITION (1970)

PRINTED IN THE UNITED STATES OF AMERICA

## PREFACE

Since its first appearance fifteen years ago, this monograph has been twice revised and updated. Meanwhile, the literature of recreational mathematics has proliferated to such an extent that instead of merely updating the original bibliography once more it seemed desirable to issue a second volume which would not only be more timely but also enlarged in scope and improved in format. The two volumes thus complement one another and provide a comprehensive coverage of the field. When used together, they should hopefully serve the reader well, whether an amateur or a professional.

—W. L. S.

*Boca Raton, Florida*
*September, 1969*

## PREFACE TO THE FIRST EDITION

The late G. H. Hardy once observed that there are few more "popular" subjects than mathematics. His contention is amply borne out by the universal interest manifested in mathematical recreations for over 2000 years, ranging from the loculus of Archimedes and the talismen magic squares of the early Chinese to the cryptanalysis and topological recreations of modern times. One need only recall how testament problems, ferrying problems, coin problems, problems of pursuit and problems of arrangements have come down through the ages, ever dressed anew, yet always the same old friends. Labyrinths, dissections, acrostics, tangrams, palindromes, and so on, are likewise virtually ageless. Hence it should occasion little surprise that an enormous body of literature has arisen in the last 300 years.

It has been my purpose to gather a considerable part of this material between the covers of one book for the convenience of students and teachers, as well as laymen and specialists. The more than 2000 entries by no means represent a complete or exhaustive compilation. But enough has been given, I hope, to be of real help. I have tried to meet the needs of almost any reader—the beginner, the dilettante, the professional scholar. Hence I have deliberately included some "popular" articles along with erudite and technical discussions; many contemporary and recent publications, as well as some of an earlier period; some that are readily accessible, and others that are to be found only in important libraries; most of them in English, some in French, German, and Italian; most of them significant, a few, somewhat superficial. In this way, it is hoped, both the neophyte and the sophisticated authority will find what they need.

The task of organizing this material yielded a more or less arbitrary classification of mathematical recreations. Occasionally, where helpful, entries have been annotated; to have commented upon each item seemed quite unnecessary, and would in any event have been prohibitive.

It would scarcely seem necessary to suggest how this guide may be used. To be sure, a number of entries listed under each of the more than 50 headings will not be available to the reader unless the facilities of a large library are at hand; yet there will almost surely be some that are accessible. In most instances the reader will have little difficulty in selecting items pertaining to a given topic: he should be guided by the title of the book or article; by the annotation, if any; by the sort of periodical, whether scholarly, popular, professional, newsy, and so on; and, to some extent, by the length of an article. Naturally, the reader's purpose, as well as his familiarity with the subject, will loom large as factors in helping him select items to be consulted. Nor should he be deterred by references in a foreign language; after all, the mathematical symbols and geometric figures are essentially the same, so that even a moderate facility in French or German often suffices.

This guide will serve as a place to begin to look for source materials. It will help the student pursuing his mathematical studies in high school or college; the mathematics club looking for program and project material; the teacher gathering human interest or motivation material; the more advanced student engaged in research; the amateur mathematician or the proverbial layman happily engaged in that most delectable of all activities—a hobby or a recreation.

May following these trails afford the reader as much pleasure as it has been for me to map them out for him.

—W. L. S.

*July 1954*

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"But leaving those of the *Body,* I shall proceed to such *Recreations* as adorn the *Mind;* of which those of the *Mathematicks* are inferior to none."

—WILLIAM LEYBOURN: *Pleasure with Profit* (1694).

## Principal Abbreviations Used

*Am.M.Mo.* = American Mathematical Monthly
*M. Gaz.* = Mathematical Gazette
*M. Mag.* = Mathematics Magazine
*M. T.* = Mathematics Teacher
*N. M. M.* = National Mathematics Magazine
*N. C. T. M.* = National Council of Teachers of Mathematics
*R. M. M.* = Recreational Mathematics Magazine
*Sci. Am.* = Scientific American
*Sci. Mo.* = Scientific Monthly
*Scrip. M.* = Scripta Mathematica
*S. S. M.* = School Science and Mathematics
*Z. M. N. U.* = Zeitschrift für Mathematischen und Naturwissenschaftlichen Unterricht

## Chapter 1

# General Works

As early as 1612 the Frenchman Claude Gaspard Bachet de Méziriac published his *Problèmes plaisans et délectables, qui se font par les nombres;* a second edition appeared in 1624. In the same year, under the *nom de plume* of Van Etten, there appeared a volume entitled *Récréations mathématiques,* the author of which was the Jesuit Jean Leurechon. General interest in such books apparently increased, for this was followed in 1630 by Claude Mydorge's *Examen du livre des récréations mathématiques et de ses problèmes.* In 1636, Daniel Schwenter's *Deliciae physicomathematical oder Mathematische und philosophische Erquickstunden* appeared posthumously, and in the years 1641-42 the Italian Jesuit Mario Bettini issued the first two volumes of his *Apiaria universae philosophiae mathematicae in quibus paradoxa et nova pleraque machinamenta exhibentur,* to be followed in 1660 by a third volume under the title of *Recreationum mathematicarum Apiaria XII novissima.* On the heels of this came the *Arithmetische Lustgarten* of Johann Mohr, published in 1665. Thirty years later we have William Leybourn's *Pleasure with Profit: Consisting of Recreations of Divers Kinds, viz., Numerical, Geometrical, Mechanical, Statical, Astronomical, Horometrical, Cryptographical, Magnetical, Automatical, Chymical, and Historical.*

At the very threshold of the 18th Century, in 1694, came Jacques Ozanam's treatise on mathematical recreations: *Récréations mathématiques et physiques.* Ozanam may be regarded as the forerunner of modern books on mathematical recreations. He drew heavily on the works of Bachet, Mydorge, and Leurechon; his own contributions were somewhat less significant. The work was later augmented and revised by Montucla, and still later rendered into English by Hutton, passing through many editions.

In more recent times, a host of illustrious names come to mind: Robert Abraham, Walter Ahrens, W. W. Rouse Ball, H. S. M. Coxeter, H. E. Dudeney, E. Fourrey, Royal V. Heath, G. Kowalewski, Maurice Kraitchik, Joseph Leeming, Walter Lietzmann, Edouard Lucas, Jerome Meyer, Geoffrey Mott-Smith, E. P. Northrop, Hubert Phillips, J. J. Proskauer, Hermann Schubert, Victor Thébault, Theodore Wolff, not to mention a score or more of others.

## 1.1 Early Twentieth Century Books—1900-1924

Ahrens, Wilhelm. *Altes und Neues aus der Unterhaltungsmathematik.* Berlin: 1918.

Ahrens, Wilhelm. *Mathematische Unterhaltungen und Spiele* (2 Vol.). Leipzig: Teubner, 1910-1918.
Extensive bibliography, Vol. 2, p. 375-431.

Ahrens, Wilhelm. *Mathematiker-Anekdoten.* Leipzig: Teubner, 1920.

Ahrens, Wilhelm. *Scherz und Ernst in der Mathematik: Geflügelte und ungeflügelte Worte.* Leipzig: 1904.

Bachman, L. *Das Schachspiel und seine historische Entwicklung.* Leipzig: 1924.

Ball, W. W. R. *Récréations mathématiques et problèmes des temps anciens et modernes* (Tran., J. Fitzpatrick). Paris: 1909-1926.

Bischoff, Dr. *Die Elemente der Kabbalah.* 1913.

Bischoff, Dr. *Mystik und Magie der Zahlen.* 1920.

Blyth, Will. *Matchstick Magic.* London: 1921.

Collins, A. Frederick. *Short Cuts in Figures.* New York: Edward J. Clode, 1916.

Czepa, A. *Mathematische Spielereien.* Stuttgart: Union, Deutsche Verlagsanstalt, 1915.

Delens, Paul A. P. *Problèmes d'arithmétiques amusantes.* Paris: Vuibert, 1914. 164 p.

Dudeney, H. E. *Amusements in Mathematics.* New York: Thomas Nelson & Sons, 1917. 258 p.

Dudeney, H. E. *The World's Best Puzzles.* Strand Magazine, 1908.

Ernst, E. *Mathematische Unterhaltungen.* Ravensburg: 1911-12.

Evans, Henry P. *The Old and the New Magic.* Chicago: Open Court Publishing Co., 1909.

Ferrol, Dr. F. *Das neue Rechnungsverfahren.* 1919.

Fitting, F. *Schubert's "Mathematische Mussestunden."* Berlin: 1924.

Fourrey, E. *Curiosités géométriques.* Paris: Vuibert et Nony, 1907, 1920. 431 p.

Genau, A. *Mathematische Überraschungen.* Arnsberg: 1913.

Ghersi, I. *Matematica dilettevole e curiosa: problemi curiosi e bizzari.* Milano: 1913. 748 p.

Hardenberg, Kuno v. *Die Lösung eines alten okulten Rätsels.* 1924.

Harris, A. V., & Waldo, L. M. *Number Games for Primary Grades.* Chicago: Beckley-Cardy, 1917.

Hellenbach, L. *Die Magie der Zahlen.* 1910.

Héraud, A. *Jeux et récréations scientifiques.* Paris: 1884-1903.

Hulisch. *Zahlenmagie in Bezug auf das Menschliche Leben.* 1910.

Ignatiev, E. J. *Mathematische Spiele. Rätsel, und Erholungen.* Petersburg: 1903.

Jones, Samuel I. *Mathematical Puzzles*. Denton, Texas: News Print, 1902. 76 p. A collection of the most amusing properties of numbers, and many of the most difficult mathematical problems with their answers.

Kowalewski, Gerhard W. H. *Mathematica delectans; ausgewählte Kapitel aus der Mathematik der Spiele in gemeinverständlicher Darstellung*. Leipzig: W. Engelmann, 1921. Heft 1. Boss-puzzle und verwandte Spiele.

Lange, M. *Das Schachspiel und seine strategischen Prinzipien*. Leipzig: Teubner, 1923.

Lean, John U. *Freaks of Figures*. Detroit: Modern Methods Publishing Co., 1907.

Lietzmann, Walter. *Trugschlüsse*. Leipzig: Teubner, 1923.

List, G. *Das Geheimnis der Runen*. 1908.

Loyd, Sam. *Cyclopedia of 5000 Puzzles, Tricks and Conundrums*. New York: Morningside Press, Franklin Bigelow Corp., 1914.

Loyd, Sam. *A Puzzle Book for Children*. Philadelphia, Pa.: David McKay Co., 1912.

Maack, Ferdinand. *Elias Artista Redivivus*. 1913.

Maack, Ferdinand. *Die heilige Mathesis*. 1924.

Maack, Ferdinand. *Raumschach*. 1909.

Maennchen, F. *Geheimnisse der Rechenkünstler*. Leipzig: Teubner, 1924.

Malone, F. *Mathematical Dexterities*. St. Louis: 1906.

Mittenzwey, L. *Mathematische Kurzweil*. 3rd edition; 6th edition. Leipzig: Klinkhardt, 1895, 1912.

Neuhaus, O. *Rechenkünste und Zahlenspiele*. 1902.

Peano, G. *Giuochi di aritmetica*. 1924.

Pfaundler, L. *Das chinesisch-japanische Go-Spiel*. Leipzig: 1908.

Rich, F. M. *The Jolly Tinker*. New York: D. Appleton & Co., 1923.

Rilly, A. *Le problème du cavalier des échecs*. 1906.

Row, T. Sundara. *Geometic Exercises in Paper-Folding*. Madras, 1893. Revised edition, Chicago: Open Court Publishing Co., 1901. 148 p.

Schubert, Hermann. *Mathematical Essays and Recreations*. (Trans. by T. J. McCormack). Chicago: Open Court Publishing Co., 1910. 149 p.

Sloane, T. O'Conner. *Rapid Arithmetic*. New York: Van Nostrand, 1922.

Teyssoneau, Ed. *100 récréations mathématiques. . . . Curiosités scientifiques*. Paris: A. L. Guyot, 1904. 185 p.

Thompson, J. E. and Sloane, T. O. *Speed and Fun with Figures*. New York: Van Nostrand, 1922. 559 p.

Weeks, Raymond. *Boys' Own Arithmetic*. New York: Dutton, 1924. 188 p.

White, W. F. *A Scrapbook of Elementary Mathematics; Notes, Recreations, Essays*. Chicago: Open Court Publishing Co., 1908. 248 p.

Wunsch, H. *Unterhaltende Rechenstunden*. Wien: Gerold, 1918.

## 1.2 Contemporary Books—From 1925 On

Abbott, Edwin A. *Flatland: A Romance of Many Dimensions*. New York: Dover Publications, 1952. 103 p.

Revised edition, after 70 years.

Abraham, Robert M. *Diversions and Pastimes: a Second Series of Winter Nights Entertainments.* New York: Dutton, 1935. 153 p.

Match and coin games; knots and strings; fun with paper; conventional puzzles.

Abraham, Robert M. *Winter Nights Entertainments.* New York: Dutton, 1933. 186 p.

Card and coin tricks; paper folding; match tricks; string games; knots.

Adams, John Paul. *Puzzles for Everybody.* New York: Avon Publishing Company, 1951. 128 p. (Paper)

Adler, Irving. *Magic House of Numbers.* New York: John Day, 1957. 128 p.

Aherns, Wilhelm. *Altes und Neues aus der Unterhaltungsmathematik.* Berlin: 1938.

Well-known classic.

Ahrens, Wilhelm. *Mathematische Spiele.* Leipzig: Teubner, 1927.

Albuquerque, Irene de. *Jogos e recreacões matemáticas.* Rio de Janeiro: Conquista, 1954.

Bakst, Aaron. *Mathematical Puzzles and Pastimes.* New York: Van Nostrand, 1954. 206 p.

Bakst, Aaron. *Mathematics, Its Magic and Mastery.* New York: Van Nostrand, 1952. 790 p.

An interesting popular exposition, with much recreational material.

Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays.* London: Macmillan, 1942. 418 p.

The granddaddy of all modern books in this field. Arithmetical and geometrical recreations; polyhedra; chessboard problems; magic squares; map-colouring; unicursal problems; Kirkman's school-girls problem; manipulate arrangements; duplication, trisection, and quadrature; calculating prodigies; cryptography and cryptanalysis.

Beer, Fritz, Pseud. "Complexus." *Fröhliches Kopfzerbrechen; 100 Aufgaben für scharfe Denker, mit einem Anhang: Lösungen und Erläuterungen.* Wien und Leipzig: M. Perles, 1934. 152 p.

Boon, Frederick C. *Puzzle Papers in Arithmetic.* London: G. Bell & Sons, 1937. 64 p.

Boucheny, G. *Curiosités et récréations mathématiques.* Paris: Larousse, 1941. 147 p. .

Brandes, Louis G. *Math Can Be Fun.* Portland, Maine: J. Weston Walch, Publisher, 1956. 200 p.

**Brown, Joseph C.** *Easy Tricks with Numbers.* Pelham, N. Y.: J. C. Brown, 1943. 48 p. (Pamphlet)

**Bruneau, A.** *Initiation à curiosités mathématiques.* Paris: Nathan, 1939. 317 p.

**Caro, Victor Eduardo.** *Los números; su historia, sus propiedades, sus mentiras y verdades.* Bogotá: Editorial Minerva, s.a., 1937. 291 p.

**Carroll, Lewis.** *Symbolic Logic, Part I, Elementary.* (4th edition, 1897, 240 p.) Newtonville, Mass.: E. C. Berkeley & Associates, 815 Washington St. Reprint, 1955.

Contains Lewis Carroll's inimitable and entertaining problems in symbolic logic.

**Clarke, L. Harwood.** *Fun with Figures.* London: William Heinemann, Ltd., 1954. 87 p.

**Collins, A. Frederick.** *Fun with Figures.* New York: Appleton-Century, 1928. 253 p.

**Congrès International de Récréation Mathématique.** *Comptes-Rendus du premier Congrès.* Bruxelles: Librairie du "Sphinx," 1935. 131 p.

**Congrès International de Récréation Mathématique.** *Comptes-Rendus du deuxieme Congrès.* Bruxelles: Librairie du "Sphinx," 1937. 103 p.

**Cuthbert, W. R.** *Days for Dates.* Alhambra, California: the author, 1944. 31 p.

**Davis, Frederick.** *Fascinating Figure Puzzles.* Burroughs Adding Machine Company, 1933. (Pamphlet)

**Degrazia, Joseph.** *Math Is Fun.* New York: Gresham Press, 1948. Emerson Books, Inc., 1954. 159 p.

Assorted puzzles, chiefly arithmetical; problems of arrangement and manipulation; cryptograms.

**Deming, A. G.** *Number Stories.* Chicago: Beckley-Cardy, 1936.

**Doyle, Joseph A.** *Wizardry in Multiplication.* Georgetown, S.C.: 1949. 28 p.

**Dudeney, H. E.** *The Canterbury Puzzles and Other Curious Problems.* New York & London: Thomas Nelson, 1908, 1949. 255 p.

A distinguished collection by a veteran puzzle expert.

**Dudeney, H. E.** *Modern Puzzles and How to Solve Them.* London: C. A. Person, 1926; New York, Stokes, 1926.

**Dudeney, H. E.** *Puzzles and Curious Problems.* London: T. Nelson & Sons, 1932.

**Dunham, David.** *Every Man a Millionaire. A Balloon Trip in the Mathematical Stratosphere of Social Relations.* New York: *Scripta Mathematica,* 1937. 97 p.

**Emde, Dr.** *Palindrome und die Satorformel.* 1925.

**Eperson, D. B.** (editor). *The Lewis Carroll Puzzle Book.* Appeal Office, 97 Crane St., Salisbury, Wiltshire, England, 1948.

**Farrugia, Vincent.** *Sharpen Your Wits.* London: Frederick Warne & Co., 1956

**Filipiak, Anthony.** *100 Puzzles.* New York: A. S. Barnes & Co., 1942. 120 p.

Excellent collection of manipulative puzzles.

**Fraser, Phyllis and Young, Edith.** *A Treasury of Games, Quizzes and Puzzles.* New York: Grossett & Dunlap, 1947. 212 p.

**Freeman, Mae and Freeman, Ira.** *Fun with Figures.* New York: Random House, 1946. 60 p.

Simple discussion of common geometric figures such as the parabola, spirals, helix, screw threads, tangrams, and such. Attractive photographs.

(The) *Friday Night Book* (a Jewish Miscellany). London: The Soncino Press, 1933.

**Friend, J. Newton.** *Numbers: Fun and Facts.* New York: Scribners, 1954. 208 p.

**Gardner, Martin.** *Fads and Fallacies.* New York: Dover Publications, Inc., 1957.

**Gardner, Martin.** *Mathematics, Magic and Mystery.* New York: Dover Publications, Inc., 1956. 176 p.

**Gilles, William F.** *The Magic and Oddities of Numbers.* New York: Vantage Press, 1953. 65 p.

**Gooda, W. G.** (Comp.). *Lloyd's Log Problem Book.* London: Lloyd's, 1944. 87 p.

**Graf, Ulrich.** *Kabarett der Mathematik; Zeichnungen von Maria-Erika Graf.* Dresden: L. Ehlermann, 1943. 96 p.

**Grumette, Murray.** *Geometricks: . . . . Album of Puzzles.* 12th revised edition. Brooklyn, N. Y.: Playcraft House, 143 East 16 St., Brooklyn, 1939.

Contains 21 cardboard tile dissection puzzles and tangrams.

**Heald, Harriet V.** *Mathematical Puzzles.* (Service Booklet #171). Washington Service Bureau, 1013 Thirteenth St., N. W., Washington, D. C.; 1941. 24 p. 10¢.

**Heath, Royal Vale.** *Mathemagic.* New York: Simon & Schuster, 1933; 138 p. Dover Publications, 1954. 126 p.

Puzzles, tricks, and games with numbers for the parlor magician.

**Hirschberg, Arthur.** *Can You Solve It?* New York: Thomas Y. Crowell, 1926, 1932. 330 p.

**Hobson, E. W., et al.** *Squaring the Circle and Other Monographs.* New York: Chelsea Publishing Co., 1953.

Four well-known essays on problems of geometry: "Squaring the Circle," by E. W. Hobson; "Ruler and Compass," by H. P. Hudson; "The Theory and Construction of Non-Differentiable Functions," by A. N. Singh, and "How To Draw a Straight Line: A Lecture on Linkages," by A. B. Kempe. An intriguing, meaty little book.

**Hunter, J. A. H.** *Fun with Figures.* Toronto: Oxford University Press, 1956. 160 p.

**Johnson, Hubert Rex.** *Recreational Exercises in Mathematics; or, "A Sheet of Paper" and Other Problems.* Washington, D. C.: 1926. 204 p.

**Jones, S. I.** *Mathematical Clubs and Recreations.* Nashville, Tenn.: S. I. Jones Co., 1122 Belvedere Drive, 1940. 256 p.

Indispensable for mathematical club programs and activities.

**Jones**, S. I. *Mathematical Nuts*. Nashville, Tenn.: S. I. Jones Co., 1936. 352 p.

A companion volume to *Mathematical Wrinkles;* contains material from trigonometry, analytics, calculus, and physics.

**Jones**, S. I. *Mathematical Wrinkles*. Nashville, Tenn.: S. I. Jones Co., 1930. 376 p.

A handbook of problems and recreations; mensuration; fourth dimension; quotations; and such.

**June**, W. M. *Stunts with Numbers, Games, and Cards*. Syracuse, N. Y.: the author, 757 Ostrom Ave., 1937. 25¢ (Pamphlet)

**Kaufman, Gerald** L. *The Book of Modern Puzzles*. New York: Dover Publications, 1954. 188 p. (Paper)

**Kaufman, Gerald** L. *Geometric Verse*. New York: Beechhurst Press, 1948. 64 p.

A unique collection of humorous verse.

**Kaufman, Gerald** L. *It's About Time*. Garden City, N. Y.: Heyday House, 1935. 168 p.

**Kerst, Bruno**. *Mathematische Spiele*. Berlin: G. Grote'sche Verlagsbuchhandlung, 1933. 90 p.

**Kinnaird, Clark** (editor). *Encyclopedia of Puzzles and Pastimes*. New York: Grosset & Dunlap, 1946. 431 p.

Contains some 2500 puzzles, many of them mathematical; includes cryptographs, dissected figures, knight's tours, logics, mazes, magic squares, palindromes, and paradoxes, as well as the usual assortment of acrostics, anagrams, crossword puzzles, quizzes, whodunits, and such.

**Kowalewski** G. *Alte und neue mathematische Spiele: Eine Einführung in die Unterhaltungsmathematik*. Leipzig: Teubner, 1930. 145 p.

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Chapter 2

# Arithmetical and Algebraic Recreations

"THE COMBINED ages of Mary and Ann are 44 years, and Mary is twice as old as Ann will be when Ann is three times as old as Mary was when Mary was three times as old as Ann. How old is Ann?" The question: *How old is Ann?*, has long since become a household byword; it is known to have been asked as early as 1789.

Many of the popular puzzles and recreations which fascinate the multitude are mathematical in nature—and a large part of these are arithmetical or algebraic. The range of subject matter, so to speak, of this large body of problems is truly amazing. In ancient and mediaeval times there were the ever-present problems of the cistern, the courier problems, the God-Greet-You problems, the lion-in-the-well problems, the time-of-day problems, and the testament problems. In mediaeval times, to be sure, emphasis shifted somewhat toward commercial problems: interest and usury, discount, insurance, coinage, exchange, weights and measures, and related matters.

In modern times, many of these old problems reappear in new guise. Of course, some new ones have been added. Like women's fashions, they appear to be subject to whimsy and caprice. Twenty-five years ago, problems of the "engineer-fireman-brakeman" type were in vogue. In turn, there would seem to be a revival of interest in a succession of classics: the monkey and the coconuts; the bumble bee flying back and forth between the radiators of two approaching automobiles; the prolific bacteria and the half-filled jar; and so on and on. At the moment of writing, the public fancy has been regaled with the egg problem of Victorian New England: Three boys, A, B, and C, went to sell their eggs. A had 10 eggs, B had 30 eggs, and C had 50 eggs. They each sold their eggs at the same rate, and received the same amount of money. How much did they sell their eggs for? No, it's not impossible.

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## 2.4 Calculating Prodigies

Lightning calculators, or so-called mathematical prodigies, have appeared from time to time, catching the public fancy. Such persons, although often illiterate, seemingly possess astonishing powers of mental computation. Most of them are relatively youthful; generally they are self-taught, and usually they do not retain their powers of calculating. Nearly all of them have had phenomenal memories for numbers. As a rule, calculating prodigies are unable to give a satisfactory explanation of their methods.

Among the more famous mental calculators were *Jedediah Buxton, Thomas Fuller, Zerah Colburn, George Bidder,* and *J. M. Zacharias Dase.* They are not to be confused with the occasional mathematicians who exhibited extraordinary aptitude for elaborate mental calculations, such as John Wallis, Andre Marie Ampere, and Carl Friedrich Gauss.

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The origins of modern number theory are to be found in ancient Greek *arithmetika,* which was a philosophy of the nature of number rather than the art of calculation; it was far more abstract than Greek geometry. Certain questions concerned the Greeks very much: the relation of primes to composite numbers; the number of primes; polygonal and solid numbers; amicable numbers; perfect numbers; *Gematria;* and such.

With the decline of Greek mathematics, progress in number theory lay dormant until about a century and a half ago. With the work of Gauss, about 1800, there began the extension of the concept of number and the generalization of arithmetic, a series of developments in which the greatest of modern mathematicians played significant roles—among them Fermat, Euler, Lagrange, Kummer, Dedekind, Kronecker, Galois, R. Lipschitz, A. Hurwitz, Emmy Noether, and L. E. Dickson.

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## 2.6 Perfect Numbers—Mersenne's Numbers

A number is said to be perfect if it equals the sum of all numbers that divide it except itself. Thus the first two perfect numbers are 6 and 28, since $6 = 1 + 2 + 3$, and $28 = 1 + 2 + 4 + 7 + 14$. Euclid was able to prove that any number of the form $2^{p-1}$ $(2^p - 1)$ is a perfect number whenever $2^p - 1$ is prime. Prime numbers of the form $2^p - 1$ are known as *Mersenne* numbers.

For upwards of 2000 years, only 12 perfect numbers were known, namely, those for which the values of $p$ in Euclid's formula are 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, and 127. In recent years, with the aid of high speed electronic computing machines, five more perfect numbers have been found, the largest, or 17th, being $2^{2280}$ $(2^{2281}-1)$.

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## 2.7 Fermat's Last Theorem

The great Fermat theorem, stating that the equation $x^n + y^n = z^n$, where $n$ is an arbitrary integer, has no integral solutions for integral values of $n$ except when $n = 1$ and $n = 2$, is as interesting today as it was some 300 years ago when first enunciated by the great master of number theory. Despite the lure of a prize of 100,000 *marks* offered shortly after the turn of the present century, all efforts to find a complete proof have thus far been fruitless. And yet Fermat claimed that "he had found a really wonderful proof, only the margin of his book was too narrow to accommodate it." To be sure, the theorem has been shown to hold for exponents below 100, but that is scarcely a mathematician's dream of success.

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Chapter 3

# Geometric Recreations

THIS RATHER broad category includes not only geometric fallacies and paradoxes, optical illusions, dissections, tangrams, and geometric constructions, but also material on regular polygons and polyhedra, tessellations, linkages, and the mechanical construction of mathematical curves. Such amusements often appeal to the eye-minded, and to those who are not particularly intrigued by numerical or algebraic puzzles.

Among some of the best known geometic fallacies are the alleged proofs that an obtuse angle equals a right angle; that every triangle is isosceles; that the length of part of a line equals the length of the whole line; and that the sum of the lengths of two sides of a triangle equals the length of the third side. These and similar proofs rarely fail to intrigue high school pupils.

As for optical illusions, the explanation generally hinges upon considerations of perspective, shading, disposition, and such, or upon purely psychological considerations. Among the most widely known optical illusions are the two equal segments with reversed arrowheads; the "Which is taller, the policeman or the little boy?"; and the "How many cubes are there—six or seven?". Optical illusions such as the last of these are the more tantalizing because they sometimes seem to "turn inside out" as you look at them.

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Geometric dissections, generally speaking, divide a given plane rectilinear figure by means of straight lines into parts which can then be reassembled to form some other preassigned configuration. Many recreations are built around such dissections. Some very well-known dissections have been applied to the proof of the Pythagorean theorem.

Tangrams go back to ancient times. They consist essentially of flat tiles or other pieces, usually seven in number, with definite shapes, such as a square, a rhombus, and five triangles. The idea is to form picture figures by suitable arrangements of the *tans*, as the pieces were called. Although an Oriental recreation, it was also known to Archimedes. His elaborate tangram consisted of 14 pieces, cut out of a rectangle whose length is twice its width—the "stomachion" of the Greeks and Romans.

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## 3.4 Regular Polygons and Polyhedrons

The elementary characteristics of regular polygons and polyhedrons were known to the ancient Greeks, who gave us the regular Platonic solids and the semi-regular Archimedian solids. But the elaborate development of the subject in modern times is scarcely 100 years old. The general theory of regular polytopes is intimately associated with several branches of higher mathematics, notably group theory, topology and *n*-dimensional geometry, not to omit its relation to the science of crystallography. A polytope is a geometrical figure bounded by portions of lines, planes, or hyperplanes; in 2-space it is a polygon, and in 3-space, a polyhedron. The study of regular polytopes is unusually fascinating. It appeals to many on the ground of sheer beauty and imagery; the mathematician cannot resist the urge to generalize; and the scientist, of course, is concerned with regular forms in Nature.

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When Lorenzo Mascheroni published his *Geometry of the Compass*, in 1797, he showed that any construction which can be executed with the straight edge and compass could also be carried out with the compass alone. Obviously, his points are not determined by the intersection of two straight lines. Furthermore, a straight line is considered as given or obtained when two points lying on it are known. Nearly 100 years later, A. Adler verified Mascheroni's claims. Adler used the idea of inversion with regard to a circle, an idea unknown to Mascheroni, having been discovered by Steiner in 1824.

Strictly speaking, Mascheroni's constructions are not usually thought of as recreations; the problems that arise, however, are not only fascinating—they make considerable demands upon one's ingenuity.

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## 3.7 Linkages—The Pantograph

The problem of transforming line motion into circular motion is simple enough, but the reverse problem, of converting circular motion into motion along a straight line, is considerably more difficult. The latter problem was of slight interest to earlier mathematicians, and only attracted widespread attention some years after the first solutions were given by Sarrus in 1853 and Peaucellier in 1864. Considerable enthusiasm in the subject of linkages developed during the last quarter of the 19th Century, stimulated largely by the work of Sylvester, Cayley, Kempe, and others, and culminating in Kempe's demonstration of the remarkable theorem that any algebraic curve can be described by a linkage. The bars of a linkage need not be straight; the only requirement is that they be plane, inextensible members. Certain linkworks are of considerable importance in mechanics and engineering.

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Chapter 4

# Assorted Recreations

FROM ONE point of view, mathematical recreations fall into two major categories: those that involve number relationships or computation, and those that depend chiefly upon the manipulation of objects. Conspicuous in the latter category we find the problem of ferrying the wolf, the goat, and the basket of cabbages across a stream (or the three couples with jealous spouses, where the boat will hold only two people); the problem of measuring out one quart of a liquid with only a 3-, 5-, and 8-quart measure available; the problem of the three coins; the twelve-coin problem; the shunting of freight cars; the Chinese ring puzzle; the problems of chains and links; the Tower of Hanoi; the Josephus problem; and the Boss Puzzle, or 15-Puzzle.

Included also among the manipulative recreations are string figures, paper-folding exercises, card tricks, chessboard problems, unicursal problems, labyrinths, and a variety of topological problems.

Because of the recent popularity of the 15-Puzzle, it merits some observations. Invented in America by Sam Lloyd in 1878, it took Europe by storm, "driving people mad." A square arrangement of 15 small square blocks numbered from 1 to 15, with room for 16 blocks, so that the 15 squares can be interchanged by sliding them about. The total number of conceivable positions is factorial 16, or almost 21 billion. It can be proved that from any given initial arrangement, only *half* of all the possible arrangements can be obtained by sliding the squares about. In the current revival of interest, the puzzle appears in dime stores, and is made of modern plastic material. Variations have also appeared—rectangular versions containing 19, 21, and 31 pieces, respectively.

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Thus one might expect that the more elaborate a map becomes, the more colors would be required if the desired condition above is to be fulfilled, but such is not the case. Curiously enough, no map has yet been constructed for which four colors would *not* be sufficient. This is very different, however, from *proving* the generalization that four colors *would suffice* for any conceivable map.

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## Chapter 5

# Magic Squares

Undoubtedly of Chinese, or at least Oriental origin, magic squares seem always to have been associated with mysticism. Through the ages they have been used in fortune telling and as talismen and amulets. Often they were associated with the symbols of the alchemist; and they played a significant role in the cabalistic writings of the Hebrews.

Although the theory of third-order squares is simple and complete, no completely general methods of construction are known, nor has a complete count of magic squares of all orders ever been made. Magic squares may be derived from a given arrangement by various transformations, such as mirror reflection, rotation through 90°, cyclic interchange of rows or columns or both, and, in the case of even-order squares, by simple interchange of opposite quarters.

In addition to ordinary magic squares, a number of interesting varieties are to be found: *bordered* squares, i.e., squares within squares; *pandiagonal* squares, i.e., squares that are magic along the broken diagonals as well as along the two main diagonals; *symmetric* squares, i.e., squares of order $n$ such that the sum of any two numbers in skewly related cells shall be constant and equal to $n^2 + 1$; magic squares of nonconsecutive numbers; *doubly-magic* squares; magic domino squares; magic cubes; magic circles; interlocked hexagons; composite squares; and so on.

The theory and construction of magic squares is related to lattice theory. Indeed, as James Byrnie Shaw has aptly said: "Latin squares, magic squares, linkages, polyhedra, crystals, groups, properties due to singularities, automorphic forms, lattices, topology, isomers, isotopes, valences, equivalences, syzygies, systems of forms, transitivity, linear dependence, functional dependence, and many other related topics all are fundamentally based on symmetries of some sort." Is it any wonder that magic squares are so fascinating?

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### Chapter 6

# The Pythagorean Relationship

THIS CELEBRATED theorem is notable, first because of the rich historical associations suggested thereby; secondly, because of the amazing variety of proofs which have been given; and thirdly, because further exploration quickly leads to interesting and perhaps unsuspected byways, such as the Golden Section, dynamic symmetry, logarithmic spirals, angle trisection, duplication of the cube, squaring the circle, determination of the value of $\pi$, the concept of the irrational number, regular and star polygons and polyhedra, theory of numbers, constructibility of angles and polygons, continued fractions, phyllotaxy, musical scales, Diaphontine equations, Heronian triangles, and Pythagorean number lore.

Two works are of particular interest: the brief monograph by Loomis, which gives over 200 proofs of the theorem, and the stimulating tract by Naber, which is unusually suggestive with respect to the ramifications of the theorem.

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A general Pythagorean triplet may be expressed as $(p, q; r)$, which means that $p$, $q$ and $r$ are distinct integers satisfying the equation $p^2 + q^2 = r^2$. If $p$, $q$ and $r$ have no factor in common, the triplet is called a *primitive* triplet.

Pythagorean triplets exhibit many interesting properties. The familiar 3, 4; 5 triplet is the only one which consists of *consecutive* positive integers. In some triplets, $p$, $q$ and $r$ form an arithmetic progression; but no Pythagorean triplet exists in which one number is a mean proportional between the other two. Again: no primitive Pythagorean triplet can contain two even numbers. Furthermore, if $(p, q; r)$ is a Pythagorean triplet, then $p$ and $q$ cannot both be odd.

Two fundamental relationships are of interest:

1. The numbers $2n + 1$, $2n(n + 1)$, and $2n^2 + 2n + 1$ form a Pythagorean triplet for every value of $n$.
2. Every primitive Pythagorean triplet $(p, q; r)$ is of the form $p = u^2 - v^2$, $q = 2uv$, $r = u^2 + v^2$, where $u$ and $v$ are relatively prime integers, one being even and the other odd, and with $u > v$.

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## Chapter 7

# Famous Problems of Antiquity

Over two thousand years ago Greek mathematicians devoted themselves to certain problems which have engaged the attention of men ever since. The many attempted solutions and the spirited controversies which these problems created through the ages served to stimulate immensely the development of mathematics, particularly algebra, equation theory, geometry, theory of numbers, group theory, and analysis.

Three of these problems are usually thought of together, namely: (a) trisecting an angle, (b) duplicating a cube, and (c) squaring a circle. As propounded by the Greeks, all three problems were to be solved by "pure Euclidean" methods—that is, by the use of compasses and the unmarked straightedge only. With this limitation—the use of straight lines and circles alone—none of these three problems can be solved. But this fact was not proved until about 1800. Nevertheless, each passing year witnesses stubborn attempts, on the part of laymen and amateurs alike, to tackle one or another of these famous "unsolved" problems and so achieve immortality.

Also of great concern to the Greeks were the famous paradoxes of Zeno. Somewhat different from the classical constructions, they presented an imposing challenge to the imagination—a challenge which, in slightly different form, plagues the mathematician even today. What is involved is nothing less than the concepts of infinity and continuity, ideas which lie not only at the roots of modern analysis, but at the very foundations of mathematics itself.

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## 7.2 Trisecting An Angle

In many ways, this has become the most famous of the three ancient problems—also the most tantalizing. It is so easy to *bisect* any angle!

Both the trigonometric and the algebraic analyses of the problem lead to an equation of the form $x^3 - 3x^3 - 2a = 0$. The question then arises: for all values of $a$, is it possible to find a root $x$ of this equation by means of compasses and straightedge alone? Modern mathematics has given an unequivocal answer: No. For it has been shown that with the straightedge and compasses together, and no other instruments, it is possible to make only those constructions which are algebraically equivalent to a finite number of operations of addition, subtraction, multiplication, division, and the extraction of real square roots involving given lengths. Yet despite this irrefutable evidence, the race of angle-trisectors, as R. C. Yates has suggested, is indeed a hardy one.

It remains to be pointed out, of course, that not a few constructions with straightedge and compasses yield remarkably close approximations for trisecting a given general angle. Some of them are so close that their discoverers often delude themselves; indeed, the mistakes in the purportedly exact constructions are often extremely difficult to detect.

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## 7.4 Squaring a Circle

In this classical problem the goal was to determine the side of a square whose area should be equal to that of a given circle. Strictly speaking, this is no more a recreation, in one sense of the term, than the trisection of an angle or the duplication of a cube. Yet the problem has a long and honorable history.

About 200 years ago it was shown that $\pi$ is incommensurable. Toward the close of the 19th Century the transcendence of $\pi$ was established. Until then the endless futile attempts to solve the problem had led to innumerable fruitful discoveries. Since then, of course, interest in the problem has all but disappeared, although the tribe of would-be circle-squarers has not yet completely vanished. It probably never will, any more than the select coterie of angle-trisectors, and those who would demolish non-Euclidean geometry as unthinkable.

The history of the problem has been well documented: for example, Montucla's *Histoire des Recherches sur la Quadrature du Cercle,* edited by

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## 7.5 History and Value of Pi ($\pi$)

The various values assigned to $\pi$ at different times and the numerous attempts to find more precise approximations constitute a fascinating story, from Archimedes' value $3\frac{1}{7} > \pi > 3\frac{10}{71}$ to Shanks' computation to 707 decimal places in 1873-74. During the 18th and 19th Centuries the number $\pi$ occupied the attention of mathematicians (including many amateurs) in connection with the problem of the quadrature of the circle. Enthusiasm for that problem diminished, however, when in 1882 Lindeman proved that $\pi$ is transcendental, although the race of circle-squarers is a hardy one.

Popular interest in the computation of the value of $\pi$ was revived late in 1949 when the ENIAC, an electronic computing machine at the U. S. Army's Ballistic Research Laboratories at Aberdeen, Md., computed $\pi$ to 2035 places certain in about 70 hours of the machine's running time.

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## 7.6 Zeno's Paradoxes

Nearly 2500 years ago, mathematicians and philosophers were greatly concerned by certain paradoxes involving the notion of the infinite. Modern mathematicians are equally puzzled by these paradoxes.

The famous paradoxes on motion, propounded by Zeno about 500 B. C., included:

1. the *Dichotomy*—that motion is impossible, because a moving object must arrive at the middle before it reaches the end;
2. *Achilles and the Tortoise*—if the tortoise is given a head start, Achilles can never overtake him;
3. the *Arrow*—which must either move where it is or move where it isn't, and so, although in flight, it is always motionless;
4. the *Stadium*—in which it appears that a given time interval is equivalent to an interval twice as great.

Beneath the apparent sophistry of these contradictions there lie subtle and elusive ideas of the most profound sort. Many explanations of the paradoxes have been offered over the years. Their meaning depends upon what interpretation is given to the logical foundations of mathematics—an area in which modern mathematicians are very far from being in agreement.

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## Chapter 8

# *Mathematical Miscellanies*

## 8.1 Mathematics in Nature

Long years before modern biology finally succumbed, as did the other physical sciences, to the relentless scrutiny of mathematical analysis, professional entomologists as well as lay naturalists observed many instances of mathematical relationships in living forms—notably shells, flowers, spider webs, honeycombs, and the like. One of the most brilliant and prolific writers in this field was the late D'Arcy Thompson. Another astute observer was the naturalist Jean Henri Fabre, "poet and prophet of the insect world," self-taught amateur mathematician, whose inimitable beauty of style is exemplified in the following passage:

"With this weird number [$\epsilon = 2.718 \ldots$] are we now stationed within the strictly defined realm of the imagination? Not at all: the catenary appears actually every time that weight and flexibility act in concert. The name is given to the curve formed by a chain suspended by two of its points which are not placed on a vertical line. It is the shape taken by a flexible cord when held at each end and relaxed; it is the line that governs the shape of a sail bellying in the wind; it is the curve of the nanny-goat's milk-bag when she returns from filling her trailing udder. And all this answers to the number $\epsilon$ . . .

"What a quantity of abstruse science for a bit of string! Let us not be surprised. A pellet of shot swinging at the end of a thread, a drop of dew trickling down a straw, a splash of water rippling under the kisses of the air, a mere trifle, after all, requires a titanic scaffolding when we wish to examine it with the eye of calculation. We need the club of Hercules to crush a fly."

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## 8.2 Machines That Think

Regardless of what the future historian may say, there is no doubt that developments during the last dozen years in the field of electronic computing machines have been little short of phenomenal. Amazingly enough, the ramifications of these unbelievably rapid developments have gone far beyond computing even of the most elaborate sort. Apparently we are on the threshold of what is yet to come in the way of *thinking machines, giant brains, logic machines*, and machines that play tit-tat-toe, gin rummy, and chess. If ever man's ingenuity and imagination have served him well, it is in this area. He has drawn upon material from symbolic logic, Boolean algebra, and binary notation, and, with the aid of the electronics engineers, has boldly synthesized mechanisms which can handle information with uncanny skill and breath-taking speed.

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## 8.3 Cryptography and Cryptanalysis

The art of writing secret messages is as old, presumably, as the human desire to convey information to certain individuals while withholding it from all others. Clearly this has utility for political and military purposes. The ability to read a secret message without having possession of the key, also a highly useful skill, has its sheer recreational and challenging aspects.

The terms *code* and *cipher* are not to be confused. A *code* is a device which requires a code dictionary to write and to read, or, more precisely, to encipher and to decipher. An encoded message is shorter than the original, or plain-text message; a few consecutive letters may represent an entire paragraph. A *cipher,* on the other hand, is as long or longer than the plain-text message.

Ciphers are of two general types: (a) *transposition ciphers,* in which the letters of the plain-text message are unchanged but their order is scrambled in some systematic manner; and (b) *substitution ciphers,* in which letters, or groups of letters, or other symbols, are substituted for letters or groups of letters of the plain text.

Thus when a bona-fide person in possession of the code book simply reverses the process of encoding a code message, he is said to decode or to decipher it. When a person having no knowledge of the key to a cipher or cryptogram "breaks" the cipher, he is said to have solved it. The art of devising secret ciphers is called *cryptography.* The art of breaking cryptograms is known as *cryptanalysis.*

Apart from the utilitarian and romantic aspects of secret messages, cryptanalysis offers an implied challenge to human ingenuity which is not easily resisted, and which intrigues many devotees to whom utility and sentiment are immaterial.

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## 8.4 Probability, Gambling, and Game Strategy

"To err is human; to forgive, divine." Man errs frequently because of the uncertainties with which he is beset. Human experience is steeped in probabilities. To be sure, some things are certain. The object dropped will surely fall to the ground. Five cards drawn from a deck at random will surely not contain five aces. Many other things are equally certain. But many more are subject to "chance," which means that we are *not* certain. In other words, we do not know; we are ignorant. It reminds one of the two perplexed weather bureau officials, one of whom suggests to the other: "Why don't we just tell them the truth—either it will rain tomorrow or it won't."

Mathematicians have at various times espoused two principal approaches to the quantitative study of probability: (a) the *subjective view,* according to which probability describes the degree of certainty or uncertainty, or the intensity of one's belief; and (b) the *statistical view,* according to which probability is regarded as the relative frequency with which an event occurs in a certain category of events; or, popularly paraphrased, "that which usually happens we say is probable; the more often it has happened, the more likely it is to happen again."

Both points of view have advantages as well as serious limitations. The calculus of probability (which draws freely upon both), has proved extremely fruitful not only to physical scientists, but to economists, sociologists, and businessmen as well. The entire institution of modern insurance rests in large part upon probability theory. In recent times, the theory of probability has seen brilliant advances such as those exemplified by sampling and quality control techniques on the one hand, and by the theory of games and strategy on the other—to cite but two of the most dramatic recent developments.

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## 8.5 The Fourth Dimension

This hoary anachronism should doubtless be left to sink into oblivion along with angle-trisection and perpetual motion. And yet—some rather intriguing matters compel our attention.

For example, there were the machinations of the charlatan Zöllner and his spiritualistic friends, who, toward the close of the 19th Century, insisted that the properties of physical space of four-dimensions admirably accounted for otherwise inexplicable psychic phenomena. Then there are the phenomena of congruence, symmetry, asymmetry, isomerism, polarization, and such, and their relation to the concept of dimensionality. Curious, also, is the connection, earnestly professed by some, between religion and "the fourth dimension." And not without interest is the use of four-dimensional configurations, or their projections, as a source of original design. Finally, we must not overlook the popular notion that, in relativity physics, time *is* the fourth dimension.

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## 8.7 Dynamic Symmetry

In the realm of art, the term symmetry generally refers to the relation of the parts of a design to the whole. Thus classical symmetry concerns the disposition of the parts of a design, or the interrelationships between *linear* *dimensions* of a design. Often based upon regular polygons or polyhedrons, it has been alluded to as Gothic symmetry, or static symmetry.

Dynamic symmetry, on the other hand, involves *proportional areas.* It is often thought of as an organic sort of symmetry, being exemplified in living organisms such as plants, flowers and leaves, and in the human figure.

The principles of dynamic symmetry were rediscovered by the late Jay Hambidge some 35 years ago. As used by Hambidge, the term is peculiarly appropriate to describe proportioning of areas, since to the Greek mathematician *δυναμει σύμμετροσ* meant "commensurable in power," particularly in a square. Thus the familiar root-*n* rectangles, the whirling square, and such, suggest the force of the term "dynamic."

Interestingly enough, the only peoples to use the principles of dynamic symmetry were the Greeks and the Egyptians. Even more interesting are the many bypaths into which the subject leads: (a) *mathematics*—the Golden section, Fibonacci numbers, continued fractions, the number system; (b) *science*—phyllotaxy, physiology, anatomy; (c) *the arts*—sculpture, ceramics, painting, architecture, design, and modern advertising and printing layout.

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## 8.8 The Golden Section

Just as the Pythagoreans first showed the relation of number to tone intervals in music, so it was also the Greeks who first claimed that there was always some law of number that was applicable to creations of nature and art, and which explained the beauty of such creations. One of the most notable of these laws is the Law of the Golden Mean, or the Golden Section.

The law appears in many forms. In geometry it arises from the division of a given line segment into mean and extreme ratio, i.e., into two parts, $a$ and $b$, such that $\frac{a}{b} = \frac{b}{a+b}$, where $a < b$. This proportion was called the "Divine Proportion" by Luca Pacioli. By setting $x = \frac{b}{a}$, we have $x^2 - x - 1 = 0$, or $x = \frac{1 \pm \sqrt{5}}{2}$. The ratio, $\frac{1 + \sqrt{5}}{2} = 1.618 \ldots$ or $\Phi$, is known as the "golden number"; the ratio $\frac{\sqrt{5} - 1}{2} = \frac{1}{\Phi} = 0.618 \ldots$, and the ratio $\frac{\sqrt{5} + 3}{2} = \Phi^2 = 2.618 \ldots$, are all intimately related.

As one pursues the ramifications of the Golden Section one encounters a variety of mathematical interrelationships: the pentagram and the regular decagon, Fibonacci numbers, continued fractions, dynamic symmetry, and so on.

The Golden Mean appears at many unexpected turns. In Nature, among various plant and animal forms, we find phyllotaxy in leaves, pentagonal symmetry in flowers and marine animals, and pentadactylism in vertebrates. In the proportions of the human body the Golden Mean is again to be found. Man has employed the same principle in the creative arts, as seen in the dynamic symmetry of early Greek vases and statues, in classical Renaissance paintings, and in various aspects of contemporary design, including "layouts" in the printing and advertising crafts. For example, the majority of people consider the most aesthetically pleasing rectangular shape that rectangle whose sides are in the approximate ratio of 8:5 $\left(\frac{1}{\Phi} = 1.618\ \ldots\right)$, or 3:5 ($\Phi = 0.618\ldots$).

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## 8.9 Mathematics and Music

"Music is the pleasure the human soul experiences from counting without being aware that it is counting."—Leibnitz

"Mathematics is the music of Reason. The musician *feels* Mathematics, the mathematician *thinks* Music."—J. J. Sylvester

"Mathematics and Music, the most sharply contrasted fields of scientific activity, are yet so related as to reveal the secret connection binding together all the activities of our mind."—Helmholz

"It is not surprising that the greatest mathematicians have again and again appealed to the arts in order to find some analogy to their own work. They have indeed found it in the most varied arts, in poetry, in painting, and in sculpture, although it would certainly seem that it is in music, the most abstract of all the arts, the art of number and of time, that we find the closest analogy."—Havelock Ellis

"La musique est au temps ce que la géométrie est à l'espace."
—Francis Warrain

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# Supplement

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